Indian Statistical Institute, Bangalore
M. Math. Second Year Second Semester
Operator Theory

Final Examination Date: 05-05-2017

Maximum marks: 100 Time: 3 hours

In the following $\mathcal{H}$ is an infinite dimensional complex separable Hilbert space. Abbreviations SOT and WOT stand for strong operator topology and weak operator topology respectively.

(1) Let x_n, y_n be vectors in the closed unit ball of $\mathcal{H}$, such that the sequence $\{\langle x_n, y_n\rangle\}_{n\geq 1}$ converges to 1 as n tends to infinity. Show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. [10]

(2) Let $\mathcal{A}$ be a unital commutative Banach algebra. For $x \in \mathcal{A}$, let $\sigma(x)$ denote the spectrum of x. Show that if ϕ is a complex homomorphism on $\mathcal{A}$, then for any $x \in \mathcal{A}$, $\phi(x) \in \sigma(x)$. Conversely if $x \in \mathcal{A}$, and $\lambda \in \sigma(x)$, then there exists a complex homomorphism ϕ such that $\phi(x) = \sigma(x)$. [15]

(3) Let $\mathcal{A}$ be a unital C^*-algebra and let $x \in \mathcal{A}$ be positive. Show that there exists a state ϕ on $\mathcal{A}$ such that $\phi(x) = \|x\|$. [15]

(4) Let $\mathcal{A}$ be the C^*-algebra of continuous functions on $\{1, 2, \dots, n\}$ with discrete topology. Show that $\phi : \mathcal{A} \to \mathbb{C}$, defined by

$$\phi(f) = \frac{1}{n}\sum_{j=1}^{n} f(j)$$

is a state. Obtain the GNS triple associated with this state. [15]

(5) Prove or disprove the following: (i) The map $(A, B) \mapsto AB$ from $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$, to $\mathcal{B}(\mathcal{H})$ is continuous in WOT. (ii) The map $A \mapsto A^*$ is continuous with WOT in $\mathcal{B}(\mathcal{H})$. (iii) The map $A \mapsto A^*$ is continuous with SOT in $\mathcal{B}(\mathcal{H})$. [15]

(6) Let $\mathcal{K} = \mathcal{H} \oplus H$. Let $A : \mathcal{K} \to \mathcal{K}$, be the operator $A(x \oplus y) = y \oplus 0$. Obtain the polar decomposition of A. [15]

(7) Let A be a positive operator on a separable Hilbert space $\mathcal{H}$, satisfying, $0 \leq A \leq I$. Let E be the spectral measure associated with A and let P be the projection $E([0, \frac{1}{2}])$. Obtain the spectral measure of $A + P$. Show that $A + P$ is a positive operator satisfying $\frac{1}{2} \leq (A + P) \leq \frac{3}{2}$. [20]